Lecture 17

In the last lecture, we have introduced Cauchy integral formula and two applications of it. One is the Liouville's theorem and another one is Morera's theorem. Today we are going to study removability of singularities and Taylor's expansion. Some applications of Taylor's expansion will also be shown.

Theorem 0.1 (Removability of Singularities)**.** *Letting Δ be a disk and f be analytic on $\Delta' = \Delta \setminus \{z_1, ..., z_n\}$, then if*

$$\lim_{z \to z_j} (z - z_j) f(z) = 0, \qquad \text{for all } j = 1, ..., n,$$

then f can be extended to be an analytic function on Δ.

Proof. The proof is trivial. By Cauchy integral formula, we know that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w,$$

where γ, z are given in figure 1 (see another file). Noticing that $f(z)$ has no definition at $z_1, ..., z_n$, but the right-hand side of the above equality has definition at $z_1, ..., z_n$. Therefore we can redefine the value of f at $z_1, ...z_n$ to be

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z_j} \, \mathrm{d}w, \qquad j = 1, ..., n$$

then f is an analytic function on Δ. $\square$

Now we begin to study Taylor's expansion. In fact there are two versions of Taylor's expansion. One is of finite type and another one is of infinite type. We take a look at the finite type first. Define

$$g_1(z) = f(z) - f(z_0),$$

where z and z_0 are shown in Figure 2 of another file. By Cauchy integral formula, we know that

$$g_1(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} - \frac{f(w)}{w - z_0} \, \mathrm{d}w = \frac{z - z_0}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)} \, \mathrm{d}w. \tag{0.1}$$

Now we study

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)} \, \mathrm{d}w.$$

Clearly by Cauchy integral formula, we know that

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^2} \, \mathrm{d}w,$$

therefore it holds

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)} \, \mathrm{d}w - f'(z_0) &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)} \, \mathrm{d}w - \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^2} \, \mathrm{d}w \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z_0} \left(\frac{1}{w - z} - \frac{1}{w - z_0} \right) \mathrm{d}w = \frac{z - z_0}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)^2} \, \mathrm{d}w \end{aligned}$$

Applying the above equality into the right-hand side of (0.1), we have

$$g_1(z) = f'(z_0)(z - z_0) + \frac{(z - z_0)^2}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)^2} \, \mathrm{d}w.$$

Hence if we define

$$g_2(z) = f(z) - f(z_0) - f'(z_0)(z - z_0),$$

then clearly

$$g_2(z) = \frac{(z - z_0)^2}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)^2} \, \mathrm{d}w.$$

Repeating all the above arguments and using induction, we can show that

Theorem 0.2 (Taylor's theorem type I)**.** *Define*

$$g_n(z) = f(z) - f(z_0) - \dots - \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

then

$$g_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)^{n+1}} \, \mathrm{d}w,$$

where z and z_0 and γ are shown in Figure 2 of another file. In other words, we can write f as

$$f(z) = f(z_0) + \dots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \frac{(z - z_0)^{n+1}}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)(w - z_0)^{n+1}} \, \mathrm{d}w.$$

Now we estimate the term $g_n(z)$. it can be shown that

$$|g_n(z)| \le \frac{1}{2\pi} \int_\gamma \frac{|f(w)|}{|w - z|} \left(\frac{|z - z_0|}{|w - z_0|} \right)^{n+1} |\mathrm{d}w|.$$

From Figure 2, it is clear that $|w - z| \ge R - |z - z_0|$ for all w on γ. Moreover since f is continuous on γ, then we can find a constant M so that $|f(w)| \le M$ for all w on γ. Therefore by the above inequality, we have

$$|g_n(z)| \le \frac{M}{2\pi} \frac{1}{R - |z - z_0|} \left(\frac{|z - z_0|}{R} \right)^{n+1} \int_\gamma |\mathrm{d}w| = \frac{MR}{R - |z - z_0|} \left(\frac{|z - z_0|}{R} \right)^{n+1}.$$

For all z in the disk $B_R(z_0)$ (see Figure 2), we have $|z - z_0| < R$. Therefore the most-right-hand side above converges to 0 as n goes to ∞. it then holds that

$$\lim_{n \to \infty} g_n(z) = 0, \qquad \text{for all } z \text{ in } B_R(z_0).$$

This shows the following theorem

Theorem 0.3 (Taylor's theorem type II)**.** *z, z_0 and γ are shown in Figure 2. Then we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

provided that f is analytic in Ω.

Now we begin to study some applications of Theorems 0.2-0.3.

Isolated Zeros

1. if z_0 is one zero of f and

$$f^{(k)}(z_0) = 0, \qquad \text{for all } k = 1, \dots,$$

then by Theorem 0.3, we have $f(z) = 0$ for all z in $B_R(z_0)$.

2. otherwise, there is k_0 such that $f'(z_0) =, ..., = f^{(k_0-1)}(z_0) = 0$ and $f^{(k_0)}(z_0) \neq 0$. Therefore by Theorem 0.2, we have

$$
\begin{aligned}
f(z) &= \frac{f^{(k_0)}(z_0)}{k_0!}(z-z_0)^{k_0} + \frac{(z-z_0)^{k_0+1}}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)(w-z_0)^{k_0+1}}\,\mathrm{d}w \\
&= \left[\frac{f^{(k_0)}(z_0)}{k_0!} + \frac{z-z_0}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)(w-z_0)^{k_0+1}}\,\mathrm{d}w\right](z-z_0)^{k_0}
\end{aligned}
$$

Letting

$$
g(z) = \frac{f^{(k_0)}(z_0)}{k_0!} + \frac{z-z_0}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)(w-z_0)^{k_0+1}}\,\mathrm{d}w,
$$

then by the fact that $f^{(k_0)}(z_0) \neq 0$, we have

$$
g(z) \neq 0, \qquad \text{for all } z \text{ in } B_\epsilon(z_0).
$$

Here ϵ is a positive number suitably small. In light that $f(z) = g(z)(z-z_0)^{k_0}$, we know that in $B_\epsilon(z_0)$, there is only one zero of f. That is z_0. There is no other zeros of f in $B_\epsilon(z_0)$.

The above arguments show that

Theorem 0.4 (Isolation of Zeros)**.** *If z_0 is one zero of f, then either $f(z) = 0$ for all z in some $B_R(z_0)$ or in a ball $B_\epsilon(z_0)$ there is just one zero (z_0) of f in $B_\epsilon(z_0)$.*

Moreover if Ω is path-connected we further have

Theorem 0.5. *If Ω is path-connected, then either $f \equiv 0$ in Ω or all zeros of f in Ω are isolated. Here Ω is path-connected means that for all z_1 and z_2 in Ω, we can find a continous path l in Ω connecting z_1 and z_2.*

Proof. if z_0 is not an isolated zeor of f, then by Theorem 0.4, we know that $f(z) = 0$ for all z in some $B_\epsilon(z_0)$. now we pick up an arbitrary point z_1 in Ω and connect z_1 with z_0 by a continuous path l (see Figure 3). Supposing that we can cover l by two balls, say B_1 and $B_2 = B_\epsilon(z_0)$. Then B_1 and B_2 must have intersection. Choosing z_2 a point in the intersection, clearly all derivatives of f at z_2 must be zero. Applying Theorem 0.2, we know that for all points in B_1, f must all equal to zero. Hence $f(z_1) = 0$. Therefore we know that if we have one zero of f which is not isolated, then f must equal to 0 in Ω. □

Isolated Poles The study of poles are similar to zeros. Supposing that z_0 is a pole of f, then we have

$$
\lim_{z\to z_0} f(z) = \infty.
$$

Now we define

$$
h(z) = 1/f(z).
$$

clearly z_0 is one zero of h. By removability of singularities, we know that h is analytic at z_0. Therefore by Theorem 0.4, we know that

Theorem 0.6. *Either $f(z) = \infty$ for all z in some $B_R(z_0)$. Or there is just one pole of f in some $B_\epsilon(z_0)$.*

Finitely many zeros of an analytic function in bounded domain Theorems 0.4-0.5 have a strightforward corollary.

Theorem 0.7. *If f is analytic in a bounded domain $\overline{\Omega}$, then f has only finitely many zeros in Ω, provided that f does not equivalently equal to 0 in Ω. Here Ω is a path-connected domain.*

Proof. If there is a sequence of different zeros of f, say $\{z_j\}$. then we have a subsequence of $\{z_j\}$, still denoted by $\{z_j\}$ so that $z_j \longrightarrow z^*$ for some z^* in $\overline{\Omega}$. Since f does not identically equal to 0 in Ω, then by Theorem 0.5, we know that there must be a tiny disk $B_\epsilon(z^*)$ such that in this tiny disk f has only one zero. That is z^*. But this is a contradiction since we know that $z_j \to z^*$ as $j \to \infty$.Therefore we cannot have infinitely many zeros in $\overline{\Omega}$ □

Moreover, Theorem 0.7 tells us that

Theorem 0.8. *If Ω is path-connected and f, g are two analytic functions on Ω, then if $f(z_j) = g(z_j)$ for a sequence of infinitely many points, then $f(z) = g(z)$ for all z in Ω.*

Proof. Define $F = f - g$, then $F(z_j) = 0$ for all $j = 1, ...$ Therefore by Theorem 0.7, we know that $F(z) = 0$ for all z in Ω which shows the Theorem 0.8. □

Laurent Series Now we assume f is a holomorphic function on the annulus $A_{R_1,R_2}(z_0)$. More precisely we define

$$A_{R_1,R_2}(z_0) = \{z : |z - z_0| \in (R_1, R_2)\}.$$

Here we assume that $R_1 < R_2$. Moreover we denote by C_1 the circle centered at z_0 with radius R_1. C_2 the circle centered at z_0 with radius R_2. The direction of C_1 and C_2 are chosen to be positive (see graph). For any given $z \in A_{R_1,R_2}(z_0)$, we can separate $A_{R_1,R_2}(z_0)$ into two parts, i.e. Region I and region II. On the common edges l_1 and l_2, the induced direction on each part I and II are drawn. Using Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{A+B+C+D} \frac{f(w)}{w-z}\,\mathrm{d}w, \qquad 0 = \frac{1}{2\pi i}\int_{E+F+G+H} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Summing the above two equalities and noticing that the integrals on the common edges B, H, D, F are cancelled, therefore we have

$$f(z) = \frac{1}{2\pi i}\int_{A+E} \frac{f(w)}{w-z}\,\mathrm{d}w + \frac{1}{2\pi i}\int_{C+G} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Clearly $A + E = C_2$ and $C + G = -C_1$, therefore it holds

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,\mathrm{d}w - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,\mathrm{d}w. \tag{0.2}$$

If $w \in C_2$, then we have $R_2 = |w - z_0| > |z - z_0|$. Therefore

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,\mathrm{d}w &= \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-z_0)-(z-z_0)}\,\mathrm{d}w = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-z_0)}\frac{1}{1-\frac{z-z_0}{w-z_0}}\,\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-z_0)}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w\right)(z-z_0)^k\end{aligned} \tag{0.3}$$

If $w \in C_1$, then we have $R_1 = |w - z_0| < |z - z_0|$. Therefore

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,\mathrm{d}w &= \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-z_0)-(z-z_0)}\,\mathrm{d}w = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{-(z-z_0)}\frac{1}{1-\frac{w-z_0}{z-z_0}}\,\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{-(z-z_0)}\sum_{k=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^k = -\sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_1} f(w)(w-z_0)^k\,\mathrm{d}w\right)(z-z_0)^{-k-1}\end{aligned} \tag{0.4}$$

If we use $l = -k - 1$ to change the indiceds in the last equality of (0.4), then we see that while k runs from 0 to ∞, l should vary from -1 to $-\infty$. Therefore (0.4) can be written as follows:

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,\mathrm{d}w \quad = \quad -\sum_{l=-1}^{-\infty}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-z_0)^{l+1}}\,\mathrm{d}w\right)(z-z_0)^l\,. \tag{0.5}$$

Applying (0.3) and (0.5) to (0.2), we get

$$f(z) = \sum_{k=-\infty}^{\infty} b_k (z-z_0)^k, \tag{0.6}$$

where

$$b_k = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w, \qquad\qquad \text{if } k \geq 0 \tag{0.7}$$

and

$$b_k = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w, \qquad\qquad \text{if } k < 0. \tag{0.8}$$

(0.6) is the so-called Laurent series. The formula (0.7)-(0.8) are used to determine the coefficients b_k. The only difference in (0.7)-(0.8) is the integration curve. If $k \geq 0$, then b_k is obtained by integration on outer circle C_2. If k is negative, b_k should be obtained by integration on inner circle C_1.